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# A shallow water model with eddy viscosity for basins with varying bottom topography

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**Abstract**
The motion of an incompressible fluid confined to a shallow basin with a varying bottom topography is considered. We introduce appropriate scalings into a three-dimensional anisotropic eddy viscosity model to derive a two-dimensional shallow water model. The global regularity of the resulting model is proved. The anisotropic form of the stress tensor in our three-dimensional eddy viscosity model plays a critical role in ensuring that the resulting shallow water model dissipates energy.

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## 1. Introduction

In this paper we will derive and analyse a system of shallow water equations that model the large-scale horizontal motion of an incompressible fluid that is confined by gravity to a fixed basin with a slowly varying bottom topography. The starting point of our derivation is a system of fluid equations for three-dimensional incompressible flow in which the effect of small-scale turbulence is modelled with eddy viscosities. We show that, for an appropriate choice of eddy viscosity model, the resulting system of shallow water equations is well posed globally in time.

Our derivation will exploit two main scaling assumptions. First, we assume that the ratio of the horizontal fluid velocity to the gravity wave speed is small, while the ratio of the length scale of the top surface height variation to the basin depth is much smaller still. This will lead to the so-called rigid-lid approximation, which we will adopt in our three-dimensional model. Second, we assume that the basin is shallow compared with the horizontal length scales of interest. In particular, the basin topography is assumed to vary on horizontal length scales

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which are much larger than the basin depth. This will lead to the smallness parameter upon which the asymptotic derivation of the shallow water model is based.

Under the same scaling assumptions, but starting from three-dimensional incompressible Euler flow, the so-called lake equations, see [4], and great lake equations, see [5], have been derived. In [9, 10] the well posedness, globally in time, of the lake equations and the great lake equations were established for weak solutions. Global well posedness results for both of these systems were established for classical solutions in [11] and for analytic solutions in [8]. The key estimates in the proofs of these results are provided by the fact that each system conserves an energy and convects a potential vorticity.

The lake and great lake systems leave out a considerable amount of physics that is crucial to properly capturing the dynamics of currents in a large lake. For example, they neglect the Coriolis force, the wind stress on the top surface, the bottom drag and the turbulence stress produced at scales that are unresolved by the model. In [13] terms were added to the lake and great lake systems to model most of these effects. Moreover, the resulting models were shown to be globally well posed, and used to simulate the currents in Lake Erie with reasonable success. Each of these additional terms was motivated by phenomenological considerations, but none was derived from a three-dimensional model.

Here we consider an incompressible fluid that is confined to a three-dimensional basin by a uniform downward gravitational field of magnitude $g$. In terms of the standard $xyz$ Cartesian coordinates with the positive $z$-axis oriented upward, the basin is defined by its orthogonal projection onto the $xy$-plane, $\Omega$, and by its bottom, $z = -b(\boldsymbol{x})$ for every $\boldsymbol{x} = (x, y) \in \Omega$. We assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$. Because no fluid enters or leaves the basin, the average level of the top surface will be independent of time. We adopt the convention that this level is at $z = 0$. We assume that $b$ is a positive, smooth function over the closure of $\Omega$. This means that the lateral walls of the basin are vertical, which is a technical assumption. We suppose that the free top surface of the fluid at time $t$ is given by $z = h(\boldsymbol{x}, t)$ and that this surface never meets the bottom (i.e. that $b(\boldsymbol{x}) + h(\boldsymbol{x}, t) > 0$). Hence, the domain occupied by the fluid, which will be denoted by $\Sigma(t)$, is given by

$$\Sigma(t) = \big\{(\boldsymbol{x}, z) \in \mathbb{R}^3 : \boldsymbol{x} \in \Omega, -b(\boldsymbol{x}) < z < h(\boldsymbol{x}, t)\big\}. \tag{1.1}$$

For this physical situation we derive a shallow water model from a system of fluid equations for three-dimensional, incompressible flow in which turbulent stress is modelled by an eddy viscosity. As is customary, the stress tensor is assumed to depend linearly on the strain-rate tensor. This linear relationship should be isotropic, however. In addition to the usual no-flux kinematic boundary conditions, we impose so-called Navier boundary conditions that relate the stress to the tangential fluid velocity. These seem to be the most appropriate to model the effect of the friction at the boundary when scales of interest are as large as or larger than the thickness of the boundary layer. Moreover, the derivation yields non-trivial leading-order dynamics with them, which is not the case with Dirichlet boundary conditions.

We will obtain a shallow water model that governs $\boldsymbol{u}(\boldsymbol{x}, t)$, the horizontal fluid velocity averaged vertically over $\boldsymbol{x} \in \Omega$ at time $t$, and $h(\boldsymbol{x}, t)$, the top surface height, by

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla}_x \boldsymbol{u} + g \boldsymbol{\nabla}_x h = b^{-1} \boldsymbol{\nabla}_x \cdot \big[ b\nu \big( \boldsymbol{\nabla}_x \boldsymbol{u} + (\boldsymbol{\nabla}_x \boldsymbol{u})^T - \boldsymbol{\nabla}_x \cdot \boldsymbol{u} \boldsymbol{I} \big) \big] - \eta \boldsymbol{u} + \boldsymbol{f} \tag{1.2}$$

$$\boldsymbol{\nabla}_x \cdot (b\boldsymbol{u}) = 0 \tag{1.3}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_{in}(\boldsymbol{x}) \tag{1.4}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{for} \quad \boldsymbol{x} \in \partial\Omega \tag{1.5}$$

$$\nu \boldsymbol{t} \cdot \big( \boldsymbol{\nabla}_x \boldsymbol{u} + (\boldsymbol{\nabla}_x \boldsymbol{u})^T \big) \cdot \boldsymbol{n} = -\beta \boldsymbol{t} \cdot \boldsymbol{u} \qquad \text{for} \quad \boldsymbol{x} \in \partial\Omega. \tag{1.6}$$

Here $\nu(\boldsymbol{x})$ and $\eta(\boldsymbol{x})$ are a positive eddy viscosity coefficient and a non-negative turbulent drag coefficient defined over $\Omega$, $\boldsymbol{I}$ is the $2 \times 2$ identity, $\boldsymbol{f}(\boldsymbol{x}, t)$ is the wind forcing defined over $\Omega \times [0, \infty)$, $\boldsymbol{n}(\boldsymbol{x})$ and $\boldsymbol{t}(\boldsymbol{x})$ are the outward unit normal and a unit tangent to $\partial\Omega$ at $\boldsymbol{x}$ and $\beta(\boldsymbol{x})$ is a non-negative turbulent boundary drag coefficient defined on $\partial\Omega$.

The outline of the paper is as follows. In the next section we introduce and render non-dimensional the three-dimensional model that serves as our starting point. We then derive our shallow water model (1.2)–(1.6) in section 3 and prove that it is globally well posed in section 4. Finally, we make some concluding remarks in section 5.

## 2. The three-dimensional model equations

In this section we give the three-dimensional incompressible Navier–Stokes model that serves as the starting point for our subsequent development. The first subsection gives the basic mass and momentum balance laws for a basin with varying bottom topography and a free top surface, and supplements them with appropriate boundary conditions, equations (2.1)–(2.7). Subsection 2.2 then gives our constitutive relation for the stress tensor in terms of the rate of strain tensor. We suppose an anisotropic linear relationship that allows for a varying preferred direction; the presence of varying bottom topography is not consistent with supposing that the vertical direction is preferred throughout the fluid. In subsection 2.3 we introduce the rigid lid and shallow water scalings and define the corresponding non-dimensional parameters. In subsection 2.4 we non-dimensionalize the three-dimensional model in terms of those parameters.

### *2.1. Three-dimensional incompressible flow*

The momentum and mass balance laws for a three-dimensional incompressible fluid motion have the general form

$$\partial_t \boldsymbol{U} + \boldsymbol{U} \cdot \boldsymbol{\nabla} \boldsymbol{U} + \boldsymbol{\nabla} p = \boldsymbol{\nabla} \cdot \mathbb{S} \tag{2.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{U} = 0 \tag{2.2}$$

$$\boldsymbol{U}|_{t=0} = \boldsymbol{U}_{in} \tag{2.3}$$

where $\boldsymbol{U} = (\boldsymbol{u}, w)$ is the three-dimensional velocity of the fluid, $\boldsymbol{\nabla} = (\boldsymbol{\nabla}_{\boldsymbol{x}}, \partial_z)$ is the three-dimensional gradient, $p$ is the so-called modified pressure, which includes the gravitational potential [3], and $\mathbb{S}$ is the stress tensor. Equations (2.1)–(2.3) would be the Euler equations if the stress tensor $\mathbb{S}$ were identically zero. Here, however, we will model the effects of small-scale turbulence. A constitutive expression for $\mathbb{S}$ in terms of $\boldsymbol{U}$ will be specified later.

Equations (2.1)–(2.3) must be supplemented by boundary conditions. Because no fluid passes through $\partial\Sigma(t)$, one must impose the kinematic boundary condition

$$\boldsymbol{U} \cdot \boldsymbol{N} = V \tag{2.4}$$

where $\boldsymbol{N}$ is the outward unit normal and $V$ is the outward normal velocity of $\partial\Sigma(t)$.

On the free top surface of $\partial\Sigma(t)$ we assume that the only external force acting on the fluid is the wind stress. In particular, we assume that the surface tension and ambient atmospheric pressure variations are negligible. The modified pressure $p$ (which includes the gravitational potential) is therefore simply set equal to the gravitational potential $gh$, while the wind stress is modelled as being parallel to the difference of the fluid velocity $\boldsymbol{U}$ and the wind velocity $\boldsymbol{U}_W$. This leads to the boundary condition

$$p\boldsymbol{N} - \mathbb{S}\boldsymbol{N} = gh\boldsymbol{N} + \beta\,(\boldsymbol{U} - \boldsymbol{U}_W)$$

where $\beta$ is a non-negative turbulent boundary drag coefficient. Because the ambient atmosphere also does not flow through the free top surface, the wind velocity $\boldsymbol{U}_W$ satisfies the kinematic condition $\boldsymbol{U}_W \cdot \boldsymbol{N} = V$. This fact taken together with (2.4) implies that $(\boldsymbol{U} - \boldsymbol{U}_W) \cdot \boldsymbol{N} = 0$, whereby the wind stress is always tangent to the free top surface. Hence, the normal component of the above boundary condition yields the so-called free boundary condition

$$p - \boldsymbol{N} \cdot \mathbb{S}\boldsymbol{N} = gh \tag{2.5}$$

while the tangential component yields the Navier boundary condition

$$-\boldsymbol{T} \cdot \mathbb{S}\boldsymbol{N} = \beta\, \boldsymbol{T} \cdot (\boldsymbol{U} - \boldsymbol{U}_W) \tag{2.6}$$

where $\boldsymbol{T}$ is any tangent vector along the free top surface of $\partial\Sigma(t)$.

On the bottom and lateral surfaces of $\partial\Sigma(t)$ we assume there is a tangential stress due to turbulent boundary drag. We impose the Navier boundary condition

$$-\boldsymbol{T} \cdot \mathbb{S}\boldsymbol{N} = \beta\, \boldsymbol{T} \cdot \boldsymbol{U} \tag{2.7}$$

where $\boldsymbol{T}$ is any tangent vector along the bottom or lateral surfaces of $\partial\Sigma(t)$ and $\beta$ is a non-negative turbulent boundary drag coefficient.

The turbulent boundary drag coefficient $\beta$ that appears in (2.6) and (2.7) is defined on $\partial\Sigma(t)$. It has units of velocity and is often modelled as being proportional to $|\boldsymbol{U}|$. We will not be so specific here, but we will assume that $\beta$ scales as $|\boldsymbol{U}|$ to leading order. Henceforth we will let $\beta_T$, $\beta_B$ and $\beta_L$ denote the function $\beta$ restricted to the top, bottom and lateral surfaces of $\partial\Sigma(t)$, respectively.

### 2.2. The stress tensor

To obtain an appropriate form for the stress tensor, we first postulate a linear constitutive relationship between $\mathbb{S}$ and the strain-rate tensor $\mathbb{D}$, which is the traceless, symmetric tensor given by

$$\mathbb{D} = \begin{pmatrix} \mathbb{D}_{\boldsymbol{xx}} & \mathbb{D}_{\boldsymbol{x}z} \\ \mathbb{D}_{z\boldsymbol{x}} & \mathbb{D}_{zz} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u} + (\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u})^T & \partial_z\boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{x}} w \\ (\partial_z\boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{x}} w)^T & 2\partial_z w \end{pmatrix}. \tag{2.8}$$

However, we observe that this relationship cannot be assumed to be isotropic because the horizontal and vertical length scales will be of different orders in a shallow water approximation. Roughly speaking, because the scale of the horizontal eddies will be much larger than that of vertical eddies, the eddy viscosity associated with horizontal eddies will be much larger than that associated with vertical eddies. Rather than consider a general linear relationship between $\mathbb{S}$ and $\mathbb{D}$, which would have 25 coefficients, we seek a simpler anisotropic relationship that will be consistent with our shallow water approximation.

Naively, one might think that the relationship should be symmetric with respect to the vertical direction. That is to say, it should be invariant under rotations about a vertical axis. In that case an elementary linear algebra calculation shows the relationship would have the form

$$\mathbb{S} = \begin{pmatrix} \mathbb{S}_{\boldsymbol{xx}} & \mathbb{S}_{\boldsymbol{x}z} \\ \mathbb{S}_{z\boldsymbol{x}} & \mathbb{S}_{zz} \end{pmatrix} = \begin{pmatrix} \nu_H\big(\mathbb{D}_{\boldsymbol{xx}} - \frac{1}{2}\operatorname{tr}(\mathbb{D}_{\boldsymbol{xx}})\boldsymbol{I}\big) + \nu_E \frac{1}{2}\operatorname{tr}(\mathbb{D}_{\boldsymbol{xx}})\boldsymbol{I} & \nu_V \mathbb{D}_{\boldsymbol{x}z} \\ \nu_V \mathbb{D}_{z\boldsymbol{x}} & \nu_E \mathbb{D}_{zz} \end{pmatrix} \tag{2.9}$$

where the positive coefficients $\nu_H(\boldsymbol{x}, z)$, $\nu_V(\boldsymbol{x}, z)$ and $\nu_E(\boldsymbol{x}, z)$ are the eddy viscosities. These coefficients have the following interpretation. The coefficient $\nu_H$ is the eddy viscosity relative to horizontal shear motion. The coefficient $\nu_V$ is the eddy viscosity relative to vertical shear motion. The coefficient $\nu_E$ can be interpreted as the bulk viscosity coefficient relative to the expansion rate in the horizontal direction (or, equivalently, to the compression rate in the vertical direction). It turns out, however, that this assumed form for the relationship given by (2.9) does not lead to a good shallow water model because the normal to the bottom of the basin is generally not vertical. While this normal is vertical to leading order, the fact that it is not to the next order enters (as we shall see below) into the derivation of the shallow water model. In contrast, the fact that the normal to the free top surface is also not generally vertical does not enter into the derivation because of the particular scalings we impose.

We strive to retain some of the simplicity of (2.9) by postulating that the relationship is symmetric with respect to some direction given by a unit vector field $\boldsymbol{\Omega}(\boldsymbol{x}, z, t)$. The constitutive relationship between $\mathbb{S}$ and $\mathbb{D}$ then takes the form

$$\mathbb{S} = \nu_H \mathbb{D}^H + \nu_V \mathbb{D}^V + \nu_E \mathbb{D}^E \tag{2.10}$$

where the strain rate tensor $\mathbb{D}$ has been decomposed as $\mathbb{D} = \mathbb{D}^H + \mathbb{D}^V + \mathbb{D}^E$ with

$$\mathbb{D}^H = \left(\mathbb{I} - \boldsymbol{\Omega}\boldsymbol{\Omega}^T\right) \mathbb{D} \left(\mathbb{I} - \boldsymbol{\Omega}\boldsymbol{\Omega}^T\right) + \tfrac{1}{2} \left(\mathbb{I} - \boldsymbol{\Omega}\boldsymbol{\Omega}^T\right) \boldsymbol{\Omega}^T \mathbb{D} \, \boldsymbol{\Omega} \tag{2.11}$$

$$\mathbb{D}^V = \left(\mathbb{I} - \boldsymbol{\Omega}\boldsymbol{\Omega}^T\right) \mathbb{D} \, \boldsymbol{\Omega}\boldsymbol{\Omega}^T + \boldsymbol{\Omega}\boldsymbol{\Omega}^T \mathbb{D} \left(\mathbb{I} - \boldsymbol{\Omega}\boldsymbol{\Omega}^T\right) \tag{2.12}$$

$$\mathbb{D}^E = \tfrac{1}{2} \left(3\boldsymbol{\Omega}\boldsymbol{\Omega}^T - \mathbb{I}\right) \boldsymbol{\Omega}^T \mathbb{D} \, \boldsymbol{\Omega} \tag{2.13}$$

where the tensor $\mathbb{I}$ is the $3 \times 3$ identity matrix. Formula (2.10) simply generalizes (2.9) to the case where the preferred direction is given by a general unit vector $\boldsymbol{\Omega}$. The coefficients $\nu_H$, $\nu_E$ and $\nu_E$ in (2.10) have interpretations as eddy viscosities similar to those they had in (2.9) but with the direction $\boldsymbol{\Omega}$ replacing the vertical direction.

All that remains is to specify the unit vector field $\boldsymbol{\Omega}$. At the bottom and top of $\Sigma(t)$ this direction has to be the normal to $\partial\Sigma(t)$. We construct a vector field throughout $\Sigma(t)$ by linearly interpolating in $z$ between the bottom and top normals whose vertical components are 1. After normalizing the resulting vector field, we obtain

$$\boldsymbol{\Omega} = \gamma(\boldsymbol{\xi}) \begin{pmatrix} \boldsymbol{\xi} \\ 1 \end{pmatrix} \tag{2.14}$$

where $\boldsymbol{\xi}(\boldsymbol{x}, z, t)$ and $\gamma(\boldsymbol{\xi})$ are given by

$$\boldsymbol{\xi} = \frac{h - z}{h + b} \boldsymbol{\nabla}_x b - \frac{z + b}{h + b} \boldsymbol{\nabla}_x h \qquad \gamma(\boldsymbol{\xi}) = \frac{1}{\sqrt{1 + |\boldsymbol{\xi}|^2}}. \tag{2.15}$$

This linear interpolation is consistent with the derivation of our shallow-water model. In fact, it had played a role earlier in the interpretation of the potential vorticity of the great lake equations [5].

This form for the stress tensor allows considerable simplification of the boundary conditions at the free top surface and the bottom. Observe from (2.14) and (2.15) that at the top surface $\boldsymbol{N} = \boldsymbol{\Omega}$ with $\boldsymbol{\xi} = -\boldsymbol{\nabla}_x h$, while at the bottom surface $\boldsymbol{N} = -\boldsymbol{\Omega}$ with $\boldsymbol{\xi} = \boldsymbol{\nabla}_x b$. The constitutive relations (2.10)–(2.13) therefore imply that at either of these boundaries the vector $\mathbb{S}\boldsymbol{N}$ decomposes into tangential and normal components as

$$\mathbb{S}\boldsymbol{N} = \nu_V \left(\mathbb{I} - \boldsymbol{N}\boldsymbol{N}^T\right) \mathbb{D}\boldsymbol{N} + \nu_E \boldsymbol{N}\boldsymbol{N}^T \mathbb{D}\boldsymbol{N}. \tag{2.16}$$

The dynamic boundary condition (2.5) at the top surface thereby becomes

$$p = gh + \nu_E \gamma(\boldsymbol{\nabla}_x h)^2 \left[\boldsymbol{\nabla}_x^T h \, \mathbb{D}_{xx} \boldsymbol{\nabla}_x h - 2\boldsymbol{\nabla}_x^T h \, \mathbb{D}_{xz} + \mathbb{D}_{zz}\right]. \tag{2.17}$$

By exploiting the fact that the family of vectors perpendicular to $\boldsymbol{N} = \pm\boldsymbol{\Omega}$ has the form $\boldsymbol{T} = (\boldsymbol{v}, -\boldsymbol{v}^T\boldsymbol{\xi})$ for some $\boldsymbol{v} \in \mathbb{R}^2$, one finds that the Navier boundary condition at the top surface (2.6) becomes

$$-\nu_V\gamma(\nabla_x h)\left[-\mathbb{D}_{xx}\nabla_x h + \left(\boldsymbol{I} - \nabla_x h\nabla_x^T h\right)\mathbb{D}_{xz} + \nabla_x h\,\mathbb{D}_{zz}\right] = \beta_T\left(\boldsymbol{I} + \nabla_x h\nabla_x^T h\right)(\boldsymbol{u} - \boldsymbol{u}_W) \tag{2.18}$$

while the Navier boundary condition at the bottom surface (2.7) becomes

$$\nu_V\gamma(\nabla_x b)\left[\mathbb{D}_{xx}\nabla_x b + \left(\boldsymbol{I} - \nabla_x b\nabla_x^T b\right)\mathbb{D}_{xz} - \nabla_x b\,\mathbb{D}_{zz}\right] = \beta_B\left(\boldsymbol{I} + \nabla_x b\nabla_x^T b\right)\boldsymbol{u}. \tag{2.19}$$

Here $\boldsymbol{I}$ denotes the $2 \times 2$ identity matrix.

### 2.3. Non-dimensionalization

The derivation of our shallow model rests on two approximations, the rigid lid and shallow water approximations, each of which is characterized by the smallness of a non-dimensional parameter. In this subsection we identify those approximations and parameters. We then introduce these approximations into our three-dimensional model through a non-dimensionalization.

*2.3.1. The rigid lid approximation.* We assume that the typical deviation $H$ of the top surface from the mean level of the fluid is much smaller than the typical depth $B$ of the fluid, i.e. that

$$H/B = \epsilon^2 \qquad \text{where} \quad \epsilon \ll 1. \tag{2.20}$$

Moreover, we demand that the hydrostatic balance enters at the leading order in the equations, thereby ensuring non-trivial dynamics. This means that the typical pressure $P$ must be of the same order of $gH$, i.e. that

$$P = gH. \tag{2.21}$$

By requiring the horizontal gradient of the pressure to be of the same order as the nonlinear term in the momentum equation, so that it enters into the momentum equation to leading order, the typical horizontal velocity $U$ is found to be given by

$$U = \epsilon\sqrt{gB}. \tag{2.22}$$

The smallness parameter $\epsilon$ can therefore be viewed as a Froude number, i.e. the ratio between the typical horizontal velocity $U$ and the typical gravity wave speed $\sqrt{gB}$.

*2.3.2. The shallow water approximation.* We also assume that the typical depth $B$ of the basin is much smaller than the typical horizontal length $L$, i.e. that

$$B/L = \delta \qquad \text{where} \quad \delta \ll 1. \tag{2.23}$$

Moreover, we demand that variations of the fluid properties over the typical horizontal length are of the same order of magnitude as the variations over the typical vertical length. This implies that vertical derivatives are larger than horizontal gradients by a factor of $\delta^{-1}$.

Finally, we require that in the incompressibility condition all terms enter at leading order. This ensures that the incompressibility of the fluid is preserved by our scaling at the leading order. Therefore, given that vertical derivatives are bigger than the horizontal gradient by a factor of $\delta^{-1}$, we must have that the typical vertical velocity $W$ must be smaller than the typical horizontal velocity $U$ for a factor of $\delta$:

$$W = \delta\, U. \tag{2.24}$$

The smallness parameter $\delta$ is clearly an aspect ratio.

*2.3.3. The non-dimensional variables.* Given that our goal is to describe the horizontal motion of currents in our basin, it is natural to non-dimensionalize the equations in terms of the horizontal length scale $L$, the horizontal velocity scale $U$ and the non-dimensional parameters $\epsilon$ and $\delta$. We consider our system on the convection time scale $T$ given by

$$T = L/U.$$

We therefore introduce the non-dimensional independent variables $\boldsymbol{x}'$, $z'$ and $t'$ by

$$\boldsymbol{x} = L\boldsymbol{x}' \qquad z = Bz' = \delta L z' \qquad t = Tt' = \frac{L}{U}t'. \tag{2.25}$$

We will now introduce non-dimensional dependent variables, also adorned with primes. Recalling (2.24), the non-dimensional horizontal and vertical velocities are defined by

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, z, t) &= U\boldsymbol{u}'(\boldsymbol{x}', z', t') \\ w(\boldsymbol{x}, z, t) &= Ww'(\boldsymbol{x}', z', t') = \delta U w'(\boldsymbol{x}', z', t'). \end{aligned} \tag{2.26}$$

These choices ensure that time derivatives are of the same order as convection terms. Recalling (2.20) and (2.23), the non-dimensional top and bottom surfaces are defined by

$$\begin{aligned} h(\boldsymbol{x}, t) &= Hh'(\boldsymbol{x}', t') = \epsilon^2 \delta L h'(\boldsymbol{x}', t') \\ b(\boldsymbol{x}) &= Bb'(\boldsymbol{x}') = \delta L b'(\boldsymbol{x}'). \end{aligned} \tag{2.27}$$

Recalling (2.21) and (2.22), the non-dimensional modified pressure is defined by

$$p(\boldsymbol{x}, z, t) = Pp'(\boldsymbol{x}', z', t') = U^2 p'(\boldsymbol{x}', z', t'). \tag{2.28}$$

We define the non-dimensional components of the stress tensor by

$$\begin{aligned} \mathbb{S}_{\boldsymbol{x}\boldsymbol{x}}(\boldsymbol{x}, z, t) &= U^2 \mathbb{S}'_{\boldsymbol{x}\boldsymbol{x}}(\boldsymbol{x}', z', t') \\ \mathbb{S}_{\boldsymbol{x}z}(\boldsymbol{x}, z, t) &= \frac{1}{\delta} U^2 \mathbb{S}'_{\boldsymbol{x}z}(\boldsymbol{x}', z', t') \\ \mathbb{S}_{zz}(\boldsymbol{x}, z, t) &= U^2 \mathbb{S}'_{zz}(\boldsymbol{x}', z', t'). \end{aligned} \tag{2.29}$$

As will soon be apparent, these choices capture the dominant contribution of each component. We define

$$\boldsymbol{\xi}(\boldsymbol{x}, z, t) = \delta\, \boldsymbol{\xi}'(\boldsymbol{x}', z', t'). \tag{2.30}$$

We define the three non-dimensional eddy viscosities by

$$\begin{aligned} \nu_H(\boldsymbol{x}, z) &= UL\nu'_H(\boldsymbol{x}', z') \\ \nu_V(\boldsymbol{x}, z) &= UL\nu'_V(\boldsymbol{x}', z') \\ \nu_E(\boldsymbol{x}, z) &= WB\nu'_E(\boldsymbol{x}', z') = \delta^2 UL\nu'_E(\boldsymbol{x}', z'). \end{aligned} \tag{2.31}$$

These choices reflect an assumption that horizontal eddies give rise to $\nu_H$ and $\nu_V$, while vertical eddies give rise to $\nu_E$. We define the three non-dimensional boundary drag coefficients by

$$\begin{aligned} \beta_T(\boldsymbol{x}, z, t) &= U\beta'_T(\boldsymbol{x}', z', t') \\ \beta_B(\boldsymbol{x}, z, t) &= U\beta'_B(\boldsymbol{x}', z', t') \\ \beta_L(\boldsymbol{x}, z, t) &= U\beta'_L(\boldsymbol{x}', z', t'). \end{aligned} \tag{2.32}$$

These choices reflect the previously stated assumption that these coefficients scale as $|\boldsymbol{U}|$. Finally, we define the non-dimensional horizontal wind velocity by

$$\boldsymbol{u}_W(\boldsymbol{x}, z, t) = U\boldsymbol{u}'_W(\boldsymbol{x}', z', t'). \tag{2.33}$$

The non-dimensional vertical wind velocity is then determined from the fact that $\boldsymbol{U}_W$ satisfies $\boldsymbol{U}_W \cdot \boldsymbol{N} = V$.

*2.3.4. The non-dimensional equations.* We now express the three-dimensional incompressible Navier–Stokes system (2.1)–(2.7) in terms of the above non-dimensional variables. For notational convenience, we henceforth drop the primes.

The motion and continuity equations, (2.1) and (2.2), are

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{u} + w\partial_z\boldsymbol{u} + \nabla_{\boldsymbol{x}} p = \nabla_{\boldsymbol{x}}\cdot\mathbb{S}_{\boldsymbol{xx}} + \frac{1}{\delta^2}\partial_z\mathbb{S}_{z\boldsymbol{x}} \tag{2.34}$$

$$\partial_t w + \boldsymbol{u}\cdot\nabla_{\boldsymbol{x}} w + w\partial_z w + \frac{1}{\delta^2}\partial_z p = \frac{1}{\delta^2}\nabla_{\boldsymbol{x}}^T\mathbb{S}_{\boldsymbol{x}z} + \frac{1}{\delta^2}\partial_z\mathbb{S}_{zz} \tag{2.35}$$

$$\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} + \partial_z w = 0. \tag{2.36}$$

Note that $\epsilon$ can enter these equations only through the constitutive relationship for $\mathbb{S}$.

The constitutive relationship (2.10) between $\mathbb{S}$ and $\mathbb{D}$ takes the form

$$\mathbb{S} = \begin{pmatrix} \mathbb{S}_{\boldsymbol{xx}} & \frac{1}{\delta}\mathbb{S}_{\boldsymbol{x}z} \\ \frac{1}{\delta}\mathbb{S}_{z\boldsymbol{x}} & \mathbb{S}_{zz} \end{pmatrix} = \nu_H\mathbb{D}^H + \nu_V\mathbb{D}^V + \delta^2\nu_E\mathbb{D}^E \tag{2.37}$$

where $\mathbb{D}^H$, $\mathbb{D}^V$ and $\mathbb{D}^E$ are again given by (2.11)–(2.13) where

$$\boldsymbol{\Omega} = \gamma(\delta\,\boldsymbol{\xi})\begin{pmatrix}\delta\,\boldsymbol{\xi}\\ 1\end{pmatrix} \qquad \mathbb{D} = \begin{pmatrix} \nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T & \frac{1}{\delta}\partial_z\boldsymbol{u} + \delta\nabla_{\boldsymbol{x}} w \\ \frac{1}{\delta}\partial_z\boldsymbol{u}^T + \delta\nabla_{\boldsymbol{x}}^T w & 2\partial_z w \end{pmatrix} \tag{2.38}$$

and

$$\boldsymbol{\xi} = \frac{\epsilon^2 h - z}{\epsilon^2 h + b}\nabla_{\boldsymbol{x}} b - \epsilon^2\frac{z+b}{\epsilon^2 h + b}\nabla_{\boldsymbol{x}} h. \tag{2.39}$$

Note that $\epsilon$ appears explicitly in only this last equation.

The non-dimensional free top boundary is given by

$$\big\{(\boldsymbol{x}, z) \in \Omega\times\{z = \epsilon^2 h(\boldsymbol{x}, t)\}\big\}. \tag{2.40}$$

Its outward normal is $\gamma(\epsilon^2\delta\nabla_{\boldsymbol{x}} h)(-\epsilon^2\delta\nabla_{\boldsymbol{x}} h, 1)$. The non-dimensional forms of the kinematic, dynamic and Navier boundary conditions (2.4), (2.17) and (2.18) are

$$w = \epsilon^2(\partial_t h + \boldsymbol{u}\cdot\nabla_{\boldsymbol{x}} h) \tag{2.41}$$

$$\begin{aligned} p = h + \delta^2\nu_E\gamma(\epsilon^2\delta\nabla_{\boldsymbol{x}} h)^2\big[&2\partial_z w - 2\epsilon^2\nabla_{\boldsymbol{x}}^T h\left(\partial_z\boldsymbol{u} + \delta^2\nabla_{\boldsymbol{x}} w\right) \\ &+\epsilon^4\delta^2\nabla_{\boldsymbol{x}}^T h\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T\right)\nabla_{\boldsymbol{x}} h\big] \end{aligned} \tag{2.42}$$

$$\begin{aligned} -\nu_V\gamma(\epsilon^2\delta\nabla_{\boldsymbol{x}} h)\big[&\left(\boldsymbol{I} - \epsilon^4\delta^2\nabla_{\boldsymbol{x}} h\nabla_{\boldsymbol{x}}^T h\right)\left(\delta^{-2}\partial_z\boldsymbol{u} + \nabla_{\boldsymbol{x}} w\right) \\ &-\epsilon^2\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T\right)\nabla_{\boldsymbol{x}} h + \epsilon^2 2\partial_z w\nabla_{\boldsymbol{x}} h\big] \\ = \beta_T&\left(\boldsymbol{I} + \epsilon^4\delta^2\nabla_{\boldsymbol{x}} h\nabla_{\boldsymbol{x}}^T h\right)(\boldsymbol{u} - \boldsymbol{u}_W). \end{aligned} \tag{2.43}$$

The non-dimensional bottom boundary is given by

$$\big\{(\boldsymbol{x}, z) \in \Omega \times \{z = -b(\boldsymbol{x})\}\big\}. \tag{2.44}$$

Its outward normal is $-\gamma(\boldsymbol{\nabla}_{\boldsymbol{x}} b)(\boldsymbol{\nabla}_{\boldsymbol{x}} b, 1)$. The non-dimensional forms of the kinematic and Navier boundary conditions, (2.4) and (2.19), are

$$w = -\boldsymbol{u} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} b \tag{2.45}$$

$$\begin{aligned}\nu_V \gamma(\delta \boldsymbol{\nabla}_{\boldsymbol{x}} b)\big[\left(\boldsymbol{I} - \delta^2 \boldsymbol{\nabla}_{\boldsymbol{x}} b \boldsymbol{\nabla}_{\boldsymbol{x}}^T b\right)\left(\delta^{-2}\partial_z \boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{x}} w\right) + \left(\boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{u} + (\boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{u})^T\right) \boldsymbol{\nabla}_{\boldsymbol{x}} b - 2\partial_z w \boldsymbol{\nabla}_{\boldsymbol{x}} b\big] \\ = \beta_B \left(\boldsymbol{I} + \delta^2 \boldsymbol{\nabla}_{\boldsymbol{x}} b \boldsymbol{\nabla}_{\boldsymbol{x}}^T b\right) \boldsymbol{u}.\end{aligned} \tag{2.46}$$

Note that $\epsilon$ does not appear explicitly in these equations.

The non-dimensional lateral boundary is given by

$$\big\{(\boldsymbol{x}, z) \in \partial\Omega \times (-b(\boldsymbol{x}), \epsilon^2 h(\boldsymbol{x}, t))\big\}. \tag{2.47}$$

Its outward normal is $(\boldsymbol{n}, 0)$, where $\boldsymbol{n}$ is the unit outward normal of $\partial\Omega$. Its tangent space is spanned by $(\boldsymbol{t}, 0)$ and $(0, 1)$, where $\boldsymbol{t}$ is a unit vector tangent to $\partial\Omega$. The non-dimensional forms of the kinematic boundary condition (2.4) and the Navier boundary conditions (2.7) for $\boldsymbol{T} = (\boldsymbol{t}, 0)$ and $\boldsymbol{T} = (0, 1)$ are

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \tag{2.48}$$

$$-\boldsymbol{t}^T \mathbb{S}_{\boldsymbol{x}\boldsymbol{x}} \boldsymbol{n} = \beta_L \boldsymbol{t}^T \boldsymbol{u} \tag{2.49}$$

$$-\mathbb{S}_{z\boldsymbol{x}} \boldsymbol{n} = \delta^2 \beta_L w. \tag{2.50}$$

The expressions for $\mathbb{S}_{\boldsymbol{x}\boldsymbol{x}}$ and $\mathbb{S}_{z\boldsymbol{x}}$ are given by the constitutive relationship described above. Note that $\epsilon$ can enter these equations only through this constitutive relationship.

## 3. Derivation of the shallow water model

The formal derivation of our shallow water model has two steps.

We first make the rigid lid approximation by setting $\epsilon = 0$ in our non-dimensional system. This approximation supposes that the deviation from the mean level of the top surface is very small. To leading order this scaling will make the top surface flat and change the corresponding boundary conditions while leaving the interior equations essentially unaffected. The resulting system is called the rigid lid model.

We then make the shallow water approximation by formally expanding solutions of the rigid lid model in powers of $\delta$. We derive our shallow water model (1.2)–(1.6) as the zeroth-order approximation. The expansion of the stress tensor is the most cumbersome part of the derivation; we have relegated the most relevant points of this calculation to an appendix.

### *3.1. The rigid lid model*

The rigid lid model is formally obtained from our non-dimensional system by simply setting $\epsilon = 0$. This has the profound effect of fixing the free top surface at $z = 0$, whereby the domain $\Sigma(t)$ for equations (1.1) becomes simply

$$\Sigma = \big\{(\boldsymbol{x}, z) \in \mathbb{R}^3 : \boldsymbol{x} \in \Omega, -b(\boldsymbol{x}) < z < 0\big\}. \tag{3.1}$$

The top boundary surface (2.40) thereby becomes

$$\big\{(\boldsymbol{x}, z) \in \Omega \times \{z = 0\}\big\} \tag{3.2}$$

with outward normal (0, 1), while the lateral boundary surface (2.47) becomes

$$\{(\boldsymbol{x}, z) \in \partial\Omega \times (-b(\boldsymbol{x}), 0)\}. \tag{3.3}$$

The bottom boundary surface (2.44) remains unchanged.

Upon setting $\epsilon = 0$ in our non-dimensional equations, the system does not change much. The biggest changes occur in the top boundary conditions (2.41)–(2.43), which become

$$w = 0 \tag{3.4}$$

$$p = h + \delta^2 \nu_E 2\partial_z w \tag{3.5}$$

$$-\nu_V[\delta^2\partial_z \boldsymbol{u} + \boldsymbol{\nabla} w] = \beta_T(\boldsymbol{u} - \boldsymbol{u}_W). \tag{3.6}$$

The only other equation that changes is the vector $\boldsymbol{\xi}$ given by (2.39), which becomes simply

$$\boldsymbol{\xi} = -\frac{z}{b}\boldsymbol{\nabla}_x b. \tag{3.7}$$

The so-called rigid lid model therefore consists of the motion and continuity equations (2.34)–(2.36), the initial condition (2.3), the top boundary conditions (3.4)–(3.6), the bottom boundary conditions (2.45) and (2.46), the lateral boundary conditions (2.48)–(2.50), and the constitutive relations (2.10)–(2.13) with $\mathbb{S}$, $\boldsymbol{\Omega}$ and $\mathbb{D}$ given by (2.37) and $\boldsymbol{\xi}$ given by (3.7).

The only place that the top surface height $h$ enters the rigid lid model is through the dynamic boundary condition (3.5). This boundary condition therefore serves only to determine $h$, and otherwise drops out of the governing dynamics. Indeed, it is called the rigid lid model because the fluid motion is governed by equations that describe a fluid confined to the fixed domain $\Sigma$. For a more complete discussion concerning the meaning of the rigid lid scaling see [5, 10].

### 3.2. *The shallow water model*

Most shallow water models derive from two assumptions: that the aspect ratio $\delta$ is small, and that the fluid is columnar (i.e. independent of the vertical variable $z$) to leading order. Here, however, because of the nature of our three-dimensional model, we only need the assumption that $\delta$ is small.

We make the shallow water approximation by formally expanding solutions of the rigid lid model in powers of $\delta$. We seek a formal solution in the form of an asymptotic series:

$$\boldsymbol{u} = \boldsymbol{u}^{(0)} + \delta^2\boldsymbol{u}^{(1)} + \cdots \tag{3.8}$$

$$w = w^{(0)} + \delta^2 w^{(1)} + \cdots \tag{3.9}$$

$$p = p^{(0)} + \delta^2 p^{(1)} + \cdots. \tag{3.10}$$

Upon inserting the above expansion into (2.34), to order $O(\delta^{-2})$ one finds that

$$\partial_z\left[\nu_V^{(0)}\partial_z\boldsymbol{u}^{(0)}\right] = 0. \tag{3.11}$$

Equation (2.35), to the leading order $O(\delta^{-2})$, gives

$$\partial_z p^{(0)} = \boldsymbol{\nabla}\cdot\left[\nu_V^{(0)}\partial_z\boldsymbol{u}^{(0)}\right] + \partial_z\left[\nu_V^{(0)}\boldsymbol{\xi}\cdot\partial_z\boldsymbol{u}^{(0)}\right]. \tag{3.12}$$

The value of $w^{(0)}$ can be found from the incompressibility condition (2.36) written at the leading order $O(1)$:

$$w^{(0)}(\boldsymbol{x}, z) = -\int_0^z \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{u}^{(0)}\,\mathrm{d}z' + w^{(0)}(\boldsymbol{x}, 0). \tag{3.13}$$

We now consider the no-flux boundary conditions (2.48), (2.45) and (3.4). To leading order they become

$$\boldsymbol{u}^{(0)} \cdot \boldsymbol{n} = 0 \qquad \text{for} \quad (\boldsymbol{x}, z) \in \partial\Omega \times [-b(\boldsymbol{x}), 0] \tag{3.14}$$

$$w^{(0)} = -\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla}_x b \qquad \text{for} \quad (\boldsymbol{x}, z) \in \Omega \times \{z = -b(\boldsymbol{x})\} \tag{3.15}$$

$$w^{(0)} = 0 \qquad \text{for} \quad (\boldsymbol{x}, z) \in \Omega \times \{z = 0\}. \tag{3.16}$$

To the leading order the dynamic boundary condition (3.5) gives

$$p^{(0)} = h^{(0)} \qquad \text{for} \quad (\boldsymbol{x}, z) \in \Omega \times \{z = 0\}. \tag{3.17}$$

The boundary conditions (2.49), (3.6) and (2.46) to leading order are

$$\boldsymbol{t} \cdot \left(\boldsymbol{\nabla}_x \boldsymbol{u}^{(0)} + (\boldsymbol{\nabla}_x \boldsymbol{u}^{(0)})^T\right) \cdot \boldsymbol{n} + \beta_L \boldsymbol{t} \cdot \boldsymbol{u}^{(0)} = 0 \qquad \text{for} \quad (\boldsymbol{x}, z) \in \partial\Omega \times [-b(\boldsymbol{x}), 0] \tag{3.18}$$

$$\partial_z \boldsymbol{u}^{(0)} = 0 \qquad \text{for} \quad (\boldsymbol{x}, z) \in \Omega \times \{z = 0\} \tag{3.19}$$

$$\partial_z \boldsymbol{u}^{(0)} = 0 \qquad \text{for} \quad (\boldsymbol{x}, z) \in \Omega \times \{z = -b(\boldsymbol{x})\}. \tag{3.20}$$

Equation (3.11), together with the two boundary conditions (3.19) and (3.20) gives that to leading order the fluid is columnar:

$$\partial_z \boldsymbol{u}^{(0)} = 0. \tag{3.21}$$

The boundary condition (3.16), together with the fact that $\boldsymbol{u}^{(0)}$ does not depend on $z$ allows us to rewrite (3.13) as

$$w^{(0)} = -z \boldsymbol{\nabla}_x \cdot \boldsymbol{u}^{(0)}. \tag{3.22}$$

The compatibility condition between this expression for $w^{(0)}$ and the boundary condition (3.15), gives the following weighted incompressibility condition for $\boldsymbol{u}^{(0)}$:

$$\boldsymbol{\nabla}_x \cdot (b \boldsymbol{u}^{(0)}) = 0. \tag{3.23}$$

Because (3.12) and (3.21) imply that $p^{(0)}$ is constant with respect to the vertical variable, the dynamical boundary condition gives

$$p^{(0)} = h^{(0)}. \tag{3.24}$$

We now pass to write (2.34) to next order O(1).

$$\begin{aligned} &\partial_t \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla}_x \boldsymbol{u}^{(0)} + \boldsymbol{\nabla}_x h^{(0)} \\ &\quad = b^{-1} \boldsymbol{\nabla}_x \cdot \left[\nu_H b \left(\boldsymbol{\nabla}_x \boldsymbol{u}^{(0)} + (\boldsymbol{\nabla}_x \boldsymbol{u}^{(0)})^T - \boldsymbol{\nabla}_x \cdot \boldsymbol{u}^{(0)} \boldsymbol{I}\right)\right] + \partial_z \left[\nu_V \partial_z \boldsymbol{u}^{(1)}\right] \\ &\quad\quad - \nu_V \left[b^{-1} \left(\boldsymbol{\nabla}_x \boldsymbol{u}^{(0)} + (\boldsymbol{\nabla}_x \boldsymbol{u}^{(0)})^T + 2 \boldsymbol{\nabla}_x \cdot \boldsymbol{u}^{(0)} \boldsymbol{I}\right) \boldsymbol{\nabla}_x b + \boldsymbol{\nabla}_x \left(\boldsymbol{\nabla}_x \cdot \boldsymbol{u}^{(0)}\right)\right]. \end{aligned} \tag{3.25}$$

To obtain a closed equation for $\boldsymbol{u}^{(0)}$ one has to get rid of the $\boldsymbol{u}^{(1)}$ in the above equation. We can accomplish this with the following procedure. First integrate (3.25) in the vertical variable $z$ from 0 to $b$, obtaining the following expression:

$$\begin{aligned} &b \left(\partial_t \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla}_x \boldsymbol{u}^{(0)} + \boldsymbol{\nabla}_x h^{(0)}\right) \\ &\quad = \boldsymbol{\nabla}_x \cdot \left[\nu_H b \left(\boldsymbol{\nabla}_x \boldsymbol{u}^{(0)} + (\boldsymbol{\nabla}_x \boldsymbol{u}^{(0)})^T - \boldsymbol{\nabla}_x \cdot \boldsymbol{u}^{(0)} \boldsymbol{I}\right)\right] + \left[\nu_V \partial_z \boldsymbol{u}^{(1)}\right]_{z=-b}^{z=0} \\ &\quad\quad - \nu_V \left[\left(\boldsymbol{\nabla}_x \boldsymbol{u}^{(0)} + (\boldsymbol{\nabla}_x \boldsymbol{u}^{(0)})^T + 2 \boldsymbol{\nabla}_x \cdot \boldsymbol{u}^{(0)} \boldsymbol{I}\right) \boldsymbol{\nabla}_x b + b \, \boldsymbol{\nabla}_x \left(\boldsymbol{\nabla}_x \cdot \boldsymbol{u}^{(0)}\right)\right]. \end{aligned} \tag{3.26}$$

Then we can find the value of $\partial_z u^{(1)}$ at the top surface and at the bottom using the boundary conditions (3.6) and (2.43), written at O(1).

$$\nu_V \partial_z \boldsymbol{u}^{(1)} - \beta_B \boldsymbol{u}^{(0)} + \nu_V \big[ b \boldsymbol{\nabla}_{\boldsymbol{x}} \left( \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(0)} \right) + \left( \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{u}^{(0)} + (\boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{u}^{(0)})^T \right) \boldsymbol{\nabla}_{\boldsymbol{x}} b + 2 \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(0)} \boldsymbol{\nabla}_{\boldsymbol{x}} b \big] = 0 \qquad (3.27)$$

for $(\boldsymbol{x}, z) \in \Omega \times \{z = -b(\boldsymbol{x})\}$

$$\nu_V \partial_z \boldsymbol{u}^{(1)} + \beta_T \left( \boldsymbol{u}^{(0)} - \boldsymbol{u}_W \right) = 0 \qquad \text{for} \quad (\boldsymbol{x}, z) \in \Omega \times \{z = 0\}. \qquad (3.28)$$

By using the above two expressions in (3.25), one obtains

$$\partial_t \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{u}^{(0)} + \boldsymbol{\nabla}_{\boldsymbol{x}} h^{(0)} = b^{-1} \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \left[ \nu_H b \big( \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{u}^{(0)} + (\boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{u}^{(0)})^T - \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{u}^{(0)} \boldsymbol{I} \big) \right] - \eta \boldsymbol{u}^{(0)} + \boldsymbol{f}. \qquad (3.29)$$

Here $\eta = \beta_B + \beta_T$, while $\boldsymbol{f} = \beta_T \boldsymbol{u}_W$ represents the wind stress. Equation (3.29) is therefore a closed equation for $\boldsymbol{u}^{(0)}$, which has to be coupled with the incompressibility condition (3.23), with the boundary conditions (3.14) and (3.18), and with an initial condition.

If one supposes that the initial condition $\boldsymbol{U}_{in}$ is compatible with the asymptotic expansion (3.8)–(3.10), i.e. after passing to non-dimensional variables, that

$$\boldsymbol{U}_{in} = \boldsymbol{U}_{in}^{(0)} + \delta^2 \boldsymbol{U}_{in}^{(1)} + \cdots \qquad (3.30)$$

with

$$\partial_z \boldsymbol{u}_{in}^{(0)} = 0 \qquad (3.31)$$

$$w_{in}^{(0)} = -\boldsymbol{u}_{in}^{(0)} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} b \qquad (3.32)$$

one obtains the form of our model equations given by equations (1.2)–(1.6).

## 4. Global well posedness

In this section we prove that our shallow water model (1.2)–(1.6) is well posed. We use Sobolev spaces with weight $b$. These are denoted $L_b^p$, $W_b^{s,p}$ and $H_b^p$ with norms $\| \cdot \|_{L_b^p}$, $\| \cdot \|_{W_b^{s,p}}$ and $\| \cdot \|_{H_b^s}$, respectively. For example,

$$\|\boldsymbol{u}\|_{L_b^p} = \left( \int_\Omega b |\boldsymbol{u}|^p \, \mathrm{d}\boldsymbol{x} \right)^{1/p}.$$

The scalar product between $\boldsymbol{u}, \boldsymbol{v} \in L_b^2$ is denoted $\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{L_b^2}$, and is defined by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{L_b^2} = \int_\Omega b \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}.$$

Moreover, we define the spaces

$$H = \left\{ \boldsymbol{u} : \boldsymbol{u} \in L_b^2, \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot (b\boldsymbol{u}) = 0, \boldsymbol{n} \cdot \boldsymbol{u} = 0, \ \boldsymbol{x} \in \partial\Omega \right\}$$

$$V = \left\{ \boldsymbol{u} : \boldsymbol{u} \in H_b^1, \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot (b\boldsymbol{u}) = 0, \boldsymbol{n} \cdot \boldsymbol{u} = 0, \ \boldsymbol{x} \in \partial\Omega \right\}.$$

**Remark 4.1.** The boundary condition and the incompressibility condition in the definition of $H$ are meant in the following weak sense:

$$\text{if} \quad \boldsymbol{u} \in H \quad \text{then} \quad \int_\Omega b \boldsymbol{u} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \phi \, \mathrm{d}\boldsymbol{x} = 0 \qquad \forall \phi \in C^\infty(\overline{\Omega}).$$

The main result of this section is the following theorem.

**Theorem 4.1.** *Let $\Omega$ be smooth. Suppose that $b(\boldsymbol{x})$, $\nu(\boldsymbol{x})$ and $\eta(\boldsymbol{x})$ are non-negative functions over $\overline{\Omega}$. Suppose, moreover, that $b$ and $\nu$ are smooth, that $b\nu \geqslant C > 0$, and that $\beta(\boldsymbol{x}) \geqslant \kappa(\boldsymbol{x})$ on $\partial\Omega$, where $\kappa(\boldsymbol{x})$ is the curvature of $\partial\Omega$ at $\boldsymbol{x}$. Let $\boldsymbol{u}_{in} \in H^2 \cap V$ and $\boldsymbol{f} \in L^2_b$.*

*Then the system (1.2)–(1.6) has a unique solution $\boldsymbol{u} \in L^\infty([0,T], H^2) \cap C([0,T], V)$. Moreover, $\partial_t \boldsymbol{u} \in L^\infty([0,T], H) \cap L^2([0,T], V)$.*

The hypothesis $\beta(\boldsymbol{x}) \geqslant \kappa(\boldsymbol{x})$ is necessary to ensure that the elliptic Stokes problem (defined in the subsection below) is coercive (see (4.2)).

The proof will be given in several steps. In subsections 4.1 and 4.2 we shall prove that the (static) Stokes problem associated with (1.2)–(1.6) is well posed. In subsection 4.3 we shall construct global (in time) weak solutions for (1.2)–(1.6). All the remaining sections are devoted to the proof that, under the hypothesis of sufficient regularity of the initial data $\boldsymbol{u}_{in}$, the weak solutions are regular and unique. In fact, through a (2D) estimate on the nonlinear term, given in subsection 4.4, in subsections 4.5 and 4.6 we shall prove the continuity in time of the solution and its uniqueness. In subsection 4.7 we shall prove a regularity result on the time derivative of the solution. This will allow us to treat the time derivative, together with the nonlinear term, as a source term: in fact, in subsections 4.8 and 4.9 we shall achieve the desired result of the regularity of the solution.

Our proof differs from the classical proof of the well posedness of the 2D Navier–Stokes equations in two respects. First we use the weighted Sobolev spaces. In fact, due to the presence of $b$ in the diffusion operator in (1.2) and in the incompressibility condition (1.3), weighted Sobolev spaces are the natural ambient spaces. The second main difference is the proof of the elliptic regularity result of the (static) Stokes problem equations (4.5)–(4.8). The remainder of our argument is classical. For the construction of the weak solution we adopted the Galerkin procedure. The proof of the regularity is achieved through the classical procedure, as can be found in [15] (see also [7]).

A problem similar to equations (1.2)–(1.6) was studied in [6], where also the vanishing viscosity limit is considered.

### *4.1. The weak elliptic Stokes problem*

For a smooth function $\alpha : \Omega \longrightarrow \mathbb{R}$, we define the bilinear form $(\!(\cdot\,,\cdot)\!)_\alpha : V \times V \longrightarrow \mathbb{R}$ by

$$(\!(\boldsymbol{u},\boldsymbol{v})\!)_\alpha = \tfrac{1}{2}\int_\Omega \alpha\left(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T - \nabla_x\cdot\boldsymbol{u}\boldsymbol{I}\right) : \left(\nabla_x \boldsymbol{v} + (\nabla_x \boldsymbol{v})^T - \nabla_x\cdot\boldsymbol{v}\boldsymbol{I}\right)\mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega \eta\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega}\alpha\beta\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}s.$$

We introduce the weak elliptic Stokes problem: given $\boldsymbol{f} \in L^2_b$ find $\boldsymbol{u} \in V$ such that

$$(\!(\boldsymbol{u},\boldsymbol{v})\!)_{b\nu} = \langle \boldsymbol{f},\boldsymbol{v}\rangle_{L^2_b} \qquad \forall \boldsymbol{v} \in V. \tag{4.1}$$

**Remark 4.2.** Suppose that $\underline{\alpha} < \alpha(\boldsymbol{x}) < \overline{\alpha}$ for some constants $\underline{\alpha}$ and $\overline{\alpha}$ such that $0 < \underline{\alpha} \leqslant 1 \leqslant \overline{\alpha} < \infty$. Then, clearly, the following inequality holds:

$$\underline{\alpha}\,(\!(\boldsymbol{u},\boldsymbol{u})\!)_1 \leqslant (\!(\boldsymbol{u},\boldsymbol{u})\!)_\alpha \leqslant \overline{\alpha}\,(\!(\boldsymbol{u},\boldsymbol{u})\!)_1.$$

Therefore, to prove the coercivity of the bilinear form $(\!(\cdot\,,\cdot)\!)_\alpha$, it is enough to prove it in the special case $\alpha \equiv 1$. This is accomplished by the following proposition.

**Proposition 4.1.** *Suppose $\beta(\boldsymbol{x}) \geqslant \kappa(\boldsymbol{x})\ \forall \boldsymbol{x} \in \partial\Omega$, where $\kappa$ is the curvature of $\partial\Omega$. Moreover, let $b$ and $\nu$ be two smooth positive functions such that $b\nu \geqslant C > 0$ with $C$ constant. Let $\boldsymbol{f} \in L^2_b$. Then there exists a unique $\boldsymbol{u} \in V$ solving the elliptic weak Stokes problem (4.1)*

**Proof.** The proof is based on the coercivity of the bilinear form $(\!(\cdot\,,\,\cdot)\!)_1$. One has

$$(\!(\boldsymbol{u},\boldsymbol{u})\!)_1 \geqslant c\,\|\boldsymbol{u}\|_{H^1_b} \tag{4.2}$$

for some constant $c$ depending only on $b$, $\nu$ and the size of $\Omega$. In fact, denoting with $u$ and $v$ the Cartesian components of $\boldsymbol{u}$, i.e. $\boldsymbol{u} = (u, v)$, one has that

$$\begin{aligned}(\!(\boldsymbol{u},\boldsymbol{u})\!)_1 &\geqslant \int_\Omega \left[\left(\partial_x u - \partial_y v\right)^2 + \left(\partial_y u + \partial_x v\right)^2\right] \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \beta|\boldsymbol{u}|^2\,\mathrm{d}s \\ &= \|\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}\|_{L^2} + \int_{\partial\Omega} (\beta-\kappa)|\boldsymbol{u}|^2\,\mathrm{d}s \qquad (4.3) \\ &\geqslant \|\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}\|_{L^2} \qquad (4.4) \\ &\geqslant c\|\boldsymbol{u}\|_{H^1_b}.\end{aligned}$$

The above estimate shows coercivity in $V$. By interpreting $\langle \boldsymbol{f}, \cdot\rangle_{L^2_b}$ as a functional on $V$ and applying the Lax–Milgram theorem, one completes the proof of the proposition. □

**Remark 4.3.** In the above estimate we have used the following identity:

$$2\int_\Omega \left[\partial_x u \partial_y v - \partial_y u \partial_x v\right] \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \kappa\,|\boldsymbol{u}|^2\,\mathrm{d}s.$$

To prove the above identity, first restrict to $\boldsymbol{u} \in C^\infty(\Omega)$:

$$\begin{aligned}2\int_\Omega \left[\partial_x u \partial_y v - \partial_y u \partial_x v\right] \mathrm{d}\boldsymbol{x} &= \int_\Omega \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \left(u\partial_y v - v\partial_y u,\, v\partial_x u - u\partial_x v\right) \mathrm{d}\boldsymbol{x} \\ &= \int_{\partial\Omega} (u\boldsymbol{t}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} v - v\boldsymbol{t}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} u)\,\mathrm{d}s \\ &= -\int_{\partial\Omega} (\boldsymbol{t}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}s = \int_{\partial\Omega} \kappa\,|\boldsymbol{u}|^2\,\mathrm{d}s.\end{aligned}$$

Then extend the result to $\boldsymbol{u} \in V$ by the usual density argument.

### *4.2. The strong Stokes problem and the elliptic estimate*

In this subsection we shall consider the following problem:

$$-b^{-1}\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \left[b\nu\left(\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u} + (\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u})^T - \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{u}\boldsymbol{I}\right)\right] + \eta\boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{x}} p = \boldsymbol{f} \tag{4.5}$$

$$\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot (b\boldsymbol{u}) = 0 \tag{4.6}$$

$$\boldsymbol{n}\cdot\boldsymbol{u} = 0 \qquad \boldsymbol{x} \in \partial\Omega \tag{4.7}$$

$$\boldsymbol{t}\cdot\left(\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u} + (\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u})^T\right)\cdot\boldsymbol{n} + \beta\boldsymbol{u}\cdot\boldsymbol{t} = 0 \qquad \boldsymbol{x} \in \partial\Omega. \tag{4.8}$$

We shall prove (under the coercivity hypothesis $\beta \geqslant \kappa$ in $\partial\Omega$) that if $\boldsymbol{f} \in L^p_b$ for some $p \in (1, \infty)$, then equations (4.5)–(4.8) admit a unique solution $\boldsymbol{u} \in W^{2,p}_b$. In fact, the following proposition is the main result of this subsection.

**Proposition 4.2.** *Suppose that $\beta(\boldsymbol{x}) \geqslant \kappa(\boldsymbol{x})$ for every $\boldsymbol{x} \in \partial\Omega$. Then, given $\boldsymbol{f} \in L^p_b$, equations (4.5)–(4.8) admit a unique solution $\boldsymbol{u} \in W^{2,p}_b$.*

In the rest of this paper we shall always suppose $\beta(\boldsymbol{x}) \geqslant \kappa(\boldsymbol{x})$ for every $\boldsymbol{x} \in \partial\Omega$. In the proof of the above proposition we shall need the following *a priori* estimate:

**Lemma 4.1.** *Let* $\boldsymbol{f} \in L^p_b$. *If* $\boldsymbol{u}$ *satisfies equations (4.5)–(4.8), then there exists a constant* $c$ *such that*

$$\|\boldsymbol{u}\|_{W^{2,p}_b} \leqslant c\big(\|\boldsymbol{f}\|_{L^p_b} + \|\boldsymbol{u}\|_{L^p_b}\big). \tag{4.9}$$

The above lemma is based on the Agmon–Douglis–Nirenberg theory. See theorem 10.5 of [1], part II. Note that the above lemma does not guarantee the existence of a solution for equations (4.5)–(4.8).

We shall also need the existence in $H^2_b$, given by the following lemma.

**Lemma 4.2.** *If* $\boldsymbol{f} \in L^2_b$ *then equations (4.5)–(4.8) admit a unique solution* $\boldsymbol{u} \in H^2_b$.

**Proof of the lemma 4.2.** Proposition 4.1 gives the existence of the $\boldsymbol{u} \in H^1_b$ solution of the weak formulation of equations (4.5)–(4.8). Now define $\omega$ as the solution of the following Poisson equation with Dirichlet boundary conditions:

$$-\nu\Delta\omega = -\boldsymbol{\nabla}_{\boldsymbol{x}} \times \big[\boldsymbol{f} + \eta\boldsymbol{u} + b^{-1}\left(\boldsymbol{\nabla}_{\boldsymbol{x}} b\nu\right) \cdot \big(\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u} + (\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u})^T - \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{u}\boldsymbol{I}\big)\big] + \big(\boldsymbol{\nabla}^{\perp}_{\boldsymbol{x}}\nu\big) \cdot \Delta\boldsymbol{u} \tag{4.10}$$

$$\omega|_{\partial\Omega} = (\beta - \kappa)\boldsymbol{u} \cdot \boldsymbol{t}. \tag{4.11}$$

Note that the above equations are the vorticity formulation of equations (4.5)–(4.8).

Given that $\boldsymbol{u} \in H^1_b$, we have that the source term in the above equations is in $H^{-1}_b(\Omega)$, while the boundary condition is in $H^{1/2}_b(\partial\Omega)$. It is a standard result in the theory of the Dirichlet problem (see, e.g., [14]) that the above problem admits a solution $\omega \in H^1_b$. Then we find a stream function $\psi$ solving the problem:

$$\Delta\psi = -b\omega + \boldsymbol{u} \cdot \boldsymbol{\nabla}^{\perp}_{\boldsymbol{x}} b$$

$$\psi|_{\partial\Omega} = 0.$$

Given that $\omega \in H^1_b$, one has that $\psi \in H^3_b$. If one defines $\boldsymbol{u}' = b^{-1}\boldsymbol{\nabla}^{\perp}_{\boldsymbol{x}}\psi$, one has that $\boldsymbol{u}' \in H^2_b$ solves equations (4.5)–(4.8) and $\boldsymbol{u}' = \boldsymbol{u}$. □

**Proof of the proposition.** To prove the existence of the solution we approximate the source term $\boldsymbol{f} \in L^p_b$ with a sequence $\boldsymbol{f}_m \in L^2_b$ converging to $\boldsymbol{f}$ in $L^p_b$. Therefore, we find a sequence $\boldsymbol{u}_m \in H^2_b$ of functions solving equations (4.5)–(4.8) with $\boldsymbol{f}_m$ as source terms. Given that $\|\boldsymbol{f}_m\|_{L^p_b}$ is bounded, and the *a priori* estimate (4.9) one has that $\boldsymbol{u}_m$ is bounded in $W^{2,p}_b \cap V$. Therefore, one can extract a subsequence weakly convergent to $\boldsymbol{u} \in W^{2,p}_b \cap V$. This convergence is enough to pass to the limit in (4.5), and one has that $\boldsymbol{u}$ solves equations (4.5)–(4.8). This concludes the proof of proposition 4.1 □

### *4.3. Construction of a global weak solution*

In this subsection we shall prove the existence of a global weak solution of equations (1.2)–(1.6). We give the following weak formulation of the above equations: given $\boldsymbol{u}_{in}$ find $\boldsymbol{u} \in L^2([0, T], V)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{u}, \boldsymbol{v}\rangle_{L^2_b} + ((\boldsymbol{u}, \boldsymbol{v}))_{b\nu} + (\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in V \tag{4.12}$$

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_{in} \tag{4.13}$$

where we have defined the following trilinear form:

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega b\boldsymbol{u} \cdot \nabla_x \boldsymbol{v} \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x}.$$

The main result will be the following proposition, where we construct through a Galerkin procedure, a solution to the above problem.

**Proposition 4.3.** *Suppose $\boldsymbol{u}_{in} \in H$. Then there exists $\boldsymbol{u} \in L^2([0, T], V)$ satisfying equations (4.12) and (4.13).*

The proof of the above proposition is standard and can be achieved in many different ways. Here we shall follow [15].

*Step 1. The Galerkin procedure.* Take a basis $\{\boldsymbol{w}_i\}$ of $V$, and define the approximate solution

$$\boldsymbol{u}^N = \sum_{i=1}^{N} g_{i,N}(t)\, \boldsymbol{w}_i.$$

As usual the coefficients $g_{i,N}(t)$ are the solutions of the set of the $N$ nonlinear ordinary differential equations (ODEs) one obtains by inserting $\boldsymbol{u}^N$ and $\boldsymbol{w}_i$, for $i = 1, \dots, N$, in (4.12). The initial conditions for these ODEs are given by projecting the initial condition $\boldsymbol{u}_{in}$ onto the space generated by the $\boldsymbol{w}_i$, for $i = 1, \dots, N$. Each of the approximations $\boldsymbol{u}^N$ exists for a time $T_n$. With the *a priori* estimates that we shall obtain in the next step, we shall also find that the existence of the $\boldsymbol{u}^N$s is global in time.

*Step 2. The a priori estimates.* The approximate solution $\boldsymbol{u}^N$ has been built in such a way to satisfy (4.12) when $\boldsymbol{v} \in span\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_N\}$. Therefore, it satisfies (4.12) with $\boldsymbol{v} = \boldsymbol{u}^N$. One can therefore obtain the following energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{u}^N\|_{L^2_b} + \big(\!\big(\boldsymbol{u}^N, \boldsymbol{u}^N\big)\!\big)_{b\nu} = 0. \tag{4.14}$$

From the above estimate one has the following lemma.

**Lemma 4.3.** *The sequence $\boldsymbol{u}^N$ is bounded in $L^\infty([0, T], H)$.*

By integrating the energy estimate (4.14) over $[0, T]$, and using the coercivity in $H^1_b$ of the form $((\cdot\, , \cdot))_{b\nu}$ given by (4.2), one obtains the following lemma.

**Lemma 4.4.** *The sequence $\boldsymbol{u}^N$ is bounded in $L^2([0, T], V)$.*

*Step 3. The estimate on the fractional derivative.* We introduce the fractional derivative with respect to time. If $f(t) \in L^2(\mathbb{R})$ we define, as usual, the fractional derivative through its action on the Fourier transform:

$$\widehat{D^\gamma_t f}(\tau) = \tau^\gamma \hat{f}(\tau)$$

where $\hat{g}(\tau)$ denotes the Fourier transform of $g$ with respect to $t$. We can therefore introduce the following spaces:

$$\mathcal{H}^\gamma\,(\mathbb{R}, V, H) = \big\{\boldsymbol{u} : \ \boldsymbol{u} \in L^2(\mathbb{R}, H) \text{ and } D^\gamma_t \boldsymbol{u} \in L^2(\mathbb{R}, H)\big\}$$

$$\mathcal{H}^\gamma_K = \{\boldsymbol{u} : \ \boldsymbol{u} \in \mathcal{H}^\gamma \text{ and } \operatorname{supp}(\boldsymbol{u}) \subset K\}$$

where $K \subseteq \mathbb{R}$ is compact.

We now extend our approximate solutions $\boldsymbol{u}^N$ to the whole real line, putting them equal to zero outside $[0, T]$, and denote by $\hat{\boldsymbol{u}}$ the Fourier transform (with respect to time) of the extended function. We use the following two lemmas.

**Lemma 4.5.** *Let $0 < \gamma < \frac{1}{4}$. Then the approximate solutions $\boldsymbol{u}^N$ belong to a bounded set of $\mathcal{H}^\gamma(\mathbb{R}, V, H)$.*

**Lemma 4.6.** *The injection*

$$\mathcal{H}^\gamma_{[0,T]}(\mathbb{R}, V, H) \hookrightarrow L^2(\mathbb{R}, H)$$

*is compact.*

The proof of lemma 4.5 can be achieved as done in [15, p 285]. Lemma 4.6 is a compactness result and the proof can be found in [15, p 274].

*Step 4. Conclusion of the proof of proposition 4.3.* Because of lemma 4.3 one has that, up to a subsequence:

$$\boldsymbol{u}^N \longrightarrow \boldsymbol{u} \qquad \text{in the weak* topology of } L^\infty([0, T], H). \tag{4.15}$$

Because of lemma 4.4 one has that, up to a subsequence:

$$\boldsymbol{u}^N \longrightarrow \boldsymbol{u} \qquad \text{in the weak topology of } L^2([0, T], V). \tag{4.16}$$

Because of lemmas 4.5 and 4.6 one has that, up to a subsequence:

$$\boldsymbol{u}^N \longrightarrow \boldsymbol{u} \qquad \text{in the strong topology of } L^2([0, T], H). \tag{4.17}$$

The convergence results given in (4.15)–(4.17) are enough to pass to the limit in the equation and the proof of proposition 4.3 is achieved.

### *4.4. The estimate on the nonlinear term*

In the rest of this section we shall use the following (2D) estimate:

**Lemma 4.7.** *Let $\Omega$ be bounded. Then, if $\boldsymbol{w} \in H^1_b$, one has*

$$\|\boldsymbol{w}\|_{L^4_b} \leqslant c\, \|\boldsymbol{w}\|^{1/2}_{L^2_b} \|\nabla_{\boldsymbol{x}} \boldsymbol{w}\|^{1/2}_{L^2_b}.$$

In what follows $\Omega$ will always be a bounded domain. The above lemma allows us to give the following estimate on the nonlinear term.

**Lemma 4.8.** *Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H^1_b$. Then it holds that*

$$\left|(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\right| \leqslant c\, \|\boldsymbol{u}\|^{1/2}_{L^2_b} \|\nabla_{\boldsymbol{x}} \boldsymbol{u}\|^{1/2}_{L^2_b} \|\nabla_{\boldsymbol{x}} \boldsymbol{v}\|_{L^2_b} \|\boldsymbol{w}\|^{1/2}_{L^2_b} \|\nabla_{\boldsymbol{x}} \boldsymbol{w}\|^{1/2}_{L^2_b}.$$

Now for $\boldsymbol{u} \in V$, define the following linear operator $B\boldsymbol{u}$:

$$B\boldsymbol{u}(\boldsymbol{w}) = (\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}).$$

By using lemma 4.8, one can easily prove the following lemma.

**Lemma 4.9.** *If $\boldsymbol{u} \in L^2([0, T], V) \cap L^\infty([0, T], H)$ then $B\boldsymbol{u} \in L^2([0, T], V')$.*

*4.5. Regularity in time of the weak solutions*

In this subsection we prove that the weak solutions constructed in proposition 4.3 are continuous almost everywhere in time. More precisely, one has the following proposition.

**Proposition 4.4.** *Suppose $\boldsymbol{u}_{in} \in H$. Then if $\boldsymbol{u}$ is a weak solution of equations (1.2)–(1.6) then $\boldsymbol{u} \in C([0,T], H)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2_{L^2_b} = 2\langle \dot{\boldsymbol{u}}, \boldsymbol{u}\rangle_{V'}. \tag{4.18}$$

**Proof.** Write (1.2) in the form

$$\partial_t \boldsymbol{u} = -\boldsymbol{\nabla}_{\boldsymbol{x}} p - \boldsymbol{u}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u} - \eta\boldsymbol{u} + b^{-1}\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\left[b\nu\left(\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u} + (\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u})^T - \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{u}\boldsymbol{I}\right)\right] \tag{4.19}$$

and interpret the right-hand side of the above equation as a functional on $V$. One has that $\boldsymbol{\nabla}_{\boldsymbol{x}} p = 0$, $\boldsymbol{u}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u} \in L^2([0,T], V')$, and that, for all $\boldsymbol{w} \in V$:

$$\int_\Omega b^{-1}\left(\eta\boldsymbol{u} - \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\left[b\nu\left(\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u} + (\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u})^T - \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{u}\boldsymbol{I}\right)\right]\right)\cdot(b\boldsymbol{w})\,\mathrm{d}\boldsymbol{x} = (\!(\boldsymbol{u},\boldsymbol{w})\!)_{b\nu}$$

which, given that $\boldsymbol{u} \in L^2([0,T], V) \cap L^\infty([0,T], H)$, is a square-integrable function of time. Therefore, one has that

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \in L^2([0,T], V'). \tag{4.20}$$

Moreover, given that

$$\boldsymbol{u} \in L^2([0,T], V) \tag{4.21}$$

by interpolating between (4.20) and (4.21), one finds that $\boldsymbol{u} \in C([0,T], H)$. Equation (4.18) then readily follows. □

*4.6. Uniqueness*

Here we prove that the weak solution is, in fact, unique.

**Proposition 4.5.** *Suppose $\boldsymbol{u}_{in} \in H$. Then the weak solution is unique.*

**Proof.** Suppose there exist two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Define $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$. One finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2_{L^2_b} = -2(\!(\boldsymbol{u},\boldsymbol{u})\!)_{b\nu} - 2(\boldsymbol{u},\boldsymbol{u}_2,\boldsymbol{u}).$$

Therefore, one has

$$\begin{aligned} -(\boldsymbol{u},\boldsymbol{u}_2,\boldsymbol{u}) &\leqslant c\|\boldsymbol{u}\|_{L^2_b}\|\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}\|_{L^2_b}\|\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}_2\|_{L^2_b} \leqslant c\|\boldsymbol{u}\|_{L^2_b}(\!(\boldsymbol{u},\boldsymbol{u})\!)\|\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}_2\|_{L^2_b} \\ &\leqslant (\!(\boldsymbol{u},\boldsymbol{u})\!)_{b\nu} + c\|\boldsymbol{u}\|^2_{L^2_b}\|\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}_2\|_{L^2_b}. \end{aligned} \tag{4.22}$$

In the first step of the above estimate we have used lemma 4.8. In the second step we have used the coercivity estimate (4.2). Therefore, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^2_{L^2_b} \leqslant c\|\boldsymbol{u}\|^2_{L^2_b}\|\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}_2\|_{L^2_b}$$

which readily gives uniqueness. □

*4.7. More regularity of the time derivative*

In the rest of this section we shall suppose that $\boldsymbol{u}_{in} \in H^2 \cap V$.

In this subsection we shall prove the following regularity result on the time derivative of the solution:

**Proposition 4.6.** *Suppose $\boldsymbol{u}_{in} \in H^2 \cap V$. Then*

$$\dot{\boldsymbol{u}} \in L^\infty([0,T], H) \cap L^2([0,T], V).$$

**Proof.** The proof is based on an *a priori* estimate on the time derivative of the approximate solution $\boldsymbol{u}^N$. If one differentiates with respect to time (4.12) and writes it with $\boldsymbol{v} = \dot{\boldsymbol{u}}^N$, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\boldsymbol{u}}^N\|_{L^2_b} = -2(\!(\dot{\boldsymbol{u}}^N, \dot{\boldsymbol{u}}^N)\!)_{bv} - 2(\dot{\boldsymbol{u}}^N, \boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}^N, \dot{\boldsymbol{u}}^N).$$

The nonlinear term can be estimated as in (4.22), by using first lemma 4.8 and then coercivity. One finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\boldsymbol{u}}^N\|_{L^2_b} + \frac{1}{2}(\!(\dot{\boldsymbol{u}}^N, \dot{\boldsymbol{u}}^N)\!)_{bv} \leqslant c\|\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}^N\|_{L^2_b}\|\dot{\boldsymbol{u}}^N\|_{L^2_b}. \tag{4.23}$$

The above estimate gives that the sequence $\dot{\boldsymbol{u}}^N$ is bounded in $L^\infty([0,T], H)$, and in $L^2([0,T], V)$. That $\dot{\boldsymbol{u}} \in L^\infty([0,T], H) \cap L^2([0,T], V)$ follows. □

*4.8. More regularity for the solution*

In this subsection we shall prove the following proposition:

**Proposition 4.7.** *Suppose $\boldsymbol{u}_{in} \in H^2 \cap V$. Then*

$$\boldsymbol{u} \in L^\infty\big([0,T], W_b^{2,4/3}\big).$$

Write the weak form of the Navier–Stokes equations in the following form:

$$(\!(\boldsymbol{u}, \boldsymbol{v})\!)_{bv} = \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{L^2_b} \qquad \forall \boldsymbol{v} \in V \tag{4.24}$$

where

$$\boldsymbol{g} = -\dot{\boldsymbol{u}} - B\boldsymbol{u}. \tag{4.25}$$

By using lemma 4.7, one obtains the estimate

$$\langle B\boldsymbol{u}, \boldsymbol{v}\rangle_{L^2_b} \leqslant \|\boldsymbol{u}\|_{L^4_b}\|\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}\|_{L^2_b}\|\boldsymbol{v}\|_{L^4_b} \leqslant \|\boldsymbol{u}\|_{L^2_b}^{1/2}\|\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}\|_{L^2_b}^{3/2}\|\boldsymbol{v}\|_{L^4_b} \leqslant c\|\boldsymbol{v}\|_{L^4_b}$$

which shows that $B\boldsymbol{u} \in (L^4_b)' = L_b^{4/3}$. The above estimate is uniform in time, therefore, $B\boldsymbol{u} \in L^\infty([0,T], L_b^{4/3})$. Given the estimate on $\dot{\boldsymbol{u}}$ given by proposition 4.6, one has that $\boldsymbol{g} \in L^\infty([0,T], L_b^{4/3})$. Using proposition 4.2, one finds that $\boldsymbol{u} \in L^\infty([0,T], W_b^{2,4/3})$.

**Remark 4.4.** The Sobolev estimate on the sup of a function in 2D, and proposition 4.7 also gives

$$\boldsymbol{u} \in L^\infty([0,T] \times \Omega).$$

*4.9. Conclusion of the proof of theorem 4.1*

The above remark 4.4 allows us to give this new estimate of the nonlinear term:

$$|(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})| \leqslant \|\boldsymbol{u}\|_{L^\infty} \|\nabla_x \boldsymbol{u}\|_{L^2_b} \|\boldsymbol{v}\|_{L^2_b}.$$

Therefore, $B\boldsymbol{u} \in L^\infty([0, T], H)$ which, with proposition 4.6, gives that $\boldsymbol{g}$ as given in (4.25) is in $L^\infty([0, T], H)$. By using proposition 4.2 again, one sees that $\boldsymbol{u} \in L^\infty([0, T], H^2)$. This concludes the proof.

## 5. Concluding remarks

In a series of papers, see [4, 5], a shallow water model was derived describing the effect of a slowly varying topography on the motion of an incompressible fluid confined on a bounded basin. The fluid moves under the effect of hydrostatic imbalance but the gravity waves were suppressed imposing that the horizontal component of the velocity is much smaller than the typical velocity of the gravity waves.

In [8–11] the mathematical structure of the mentioned model was investigated, showing the global well posedness of the model equations. In all of these papers the fluid was assumed to be inviscid.

In this paper we have restored the effect of the viscous stresses. Eddy viscosity has been introduced and the Navier boundary conditions (instead of the more usual no-slip boundary conditions) have been imposed. We have carried a formal asymptotic expansion (the same as in [4, 5]) up to leading order, and derived model equations that we have proved to be globally well posed.

The following problems remain open. First, it is natural to ask to rigorously justify the formal asymptotic expansion we have used in this paper, thus proving that our model equations describe, to a good approximation, the actual dynamics of a fluid confined in a shallow basin. This was done for the step going from the three-dimensional rigid lid equations to the lake equations in [12]. In a slightly different setting this problem has been addressed in [2].

In [5] the asymptotic procedure was carried up to second order in the aspect ratio. It would be of interest to see whether this can also be accomplished in the case of the present study.

Finally, it would be natural to investigate the zero-viscosity limit of our model equations and prove that the solutions converge to the solution of the model derived in [4].

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## Appendix

One sees from (2.11)–(2.13) that

$$\mathbb{D}^H = \mathbb{D} - \left(\boldsymbol{\Omega}\boldsymbol{\Omega}^T\mathbb{D} + \mathbb{D}\boldsymbol{\Omega}\boldsymbol{\Omega}^T\right) + \tfrac{1}{2}\left(\mathbb{I} - \boldsymbol{\Omega}\boldsymbol{\Omega}^T\right)\left(\boldsymbol{\Omega}^T\mathbb{D}\,\boldsymbol{\Omega}\right) \tag{A.1}$$

$$\mathbb{D}^V = \left(\boldsymbol{\Omega}\boldsymbol{\Omega}^T\mathbb{D} + \mathbb{D}\boldsymbol{\Omega}\boldsymbol{\Omega}^T\right) - 2\boldsymbol{\Omega}\boldsymbol{\Omega}^T\left(\boldsymbol{\Omega}^T\mathbb{D}\,\boldsymbol{\Omega}\right) \tag{A.2}$$

$$\mathbb{D}^E = \tfrac{1}{2}\left(3\boldsymbol{\Omega}\boldsymbol{\Omega}^T - \mathbb{I}\right)\left(\boldsymbol{\Omega}^T\mathbb{D}\,\boldsymbol{\Omega}\right). \tag{A.3}$$

One sees from (2.38) that

$$\frac{1}{\gamma^2}\boldsymbol{\Omega}^T\mathbb{D}\boldsymbol{\Omega} = \mathbb{D}_{zz} + \delta 2\boldsymbol{\xi}^T\mathbb{D}_{\boldsymbol{x}z} + \delta^2\boldsymbol{\xi}^T\mathbb{D}_{\boldsymbol{x}\boldsymbol{x}}\boldsymbol{\xi}$$
$$= 2\left(\partial_z w + \boldsymbol{\xi}^T\partial_z\boldsymbol{u}\right) + \delta^2\left(2\boldsymbol{\xi}^T\nabla_{\boldsymbol{x}}w + \boldsymbol{\xi}^T\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T\right)\boldsymbol{\xi}\right)$$

and that

$$\frac{1}{\gamma^2}\left(\mathbb{D}\boldsymbol{\Omega}\boldsymbol{\Omega}^T + \boldsymbol{\Omega}\boldsymbol{\Omega}^T\mathbb{D}\right) = \begin{pmatrix} \begin{array}{l}\delta(\mathbb{D}_{\boldsymbol{x}z}\boldsymbol{\xi}^T + \boldsymbol{\xi}\mathbb{D}_{z\boldsymbol{x}})\\ \quad +\delta^2(\mathbb{D}_{\boldsymbol{x}\boldsymbol{x}}\boldsymbol{\xi}\boldsymbol{\xi}^T + \boldsymbol{\xi}\boldsymbol{\xi}^T\mathbb{D}_{\boldsymbol{x}\boldsymbol{x}})\end{array} & \begin{array}{l}\mathbb{D}_{\boldsymbol{x}z} + \delta^2\boldsymbol{\xi}\boldsymbol{\xi}^T\mathbb{D}_{\boldsymbol{x}z}\\ \quad +\delta(\mathbb{D}_{\boldsymbol{x}\boldsymbol{x}}\boldsymbol{\xi} + \mathbb{D}_{zz}\boldsymbol{\xi})\end{array} \\ \begin{array}{l}\mathbb{D}_{z\boldsymbol{x}} + \delta^2\mathbb{D}_{z\boldsymbol{x}}\boldsymbol{\xi}\boldsymbol{\xi}^T\\ \quad +\delta(\boldsymbol{\xi}^T\mathbb{D}_{\boldsymbol{x}\boldsymbol{x}} + \mathbb{D}_{zz}\boldsymbol{\xi}^T)\end{array} & 2\mathbb{D}_{zz} + \delta 2\mathbb{D}_{z\boldsymbol{x}}\boldsymbol{\xi} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{array}{l}\partial_z\boldsymbol{u}\boldsymbol{\xi}^T + \boldsymbol{\xi}\partial_z\boldsymbol{u}^T\\ \quad +\delta^2\big[\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T\right)\boldsymbol{\xi}\boldsymbol{\xi}^T\\ \quad +\boldsymbol{\xi}\boldsymbol{\xi}^T\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T\right)\\ \quad +\left(\nabla_{\boldsymbol{x}}w\boldsymbol{\xi}^T + \boldsymbol{\xi}\nabla_{\boldsymbol{x}}^T w\right)\big]\end{array} & \begin{array}{l}\delta^{-1}\partial_z\boldsymbol{u}\\ \quad +\delta\big[\nabla_{\boldsymbol{x}}w + \boldsymbol{\xi}\boldsymbol{\xi}^T\partial_z\boldsymbol{u} + 2\partial_z w\boldsymbol{\xi}\\ \quad +\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T\right)\boldsymbol{\xi}\big]\\ \quad +\delta^3\left[\boldsymbol{\xi}\boldsymbol{\xi}^T\nabla_{\boldsymbol{x}}w\right]\end{array} \\ \begin{array}{l}\delta^{-1}\partial_z\boldsymbol{u}^T\\ \quad +\delta\big[\nabla_{\boldsymbol{x}}^T w + \partial_z\boldsymbol{u}^T\boldsymbol{\xi}\boldsymbol{\xi}^T + 2\partial_z w\boldsymbol{\xi}^T\\ \quad +\boldsymbol{\xi}^T\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T\right)\big]\\ \quad +\delta^3\left[\nabla_{\boldsymbol{x}}^T w\boldsymbol{\xi}\boldsymbol{\xi}^T\right]\end{array} & \begin{array}{l}4\partial_z w + 2\boldsymbol{\xi}^T\partial_z\boldsymbol{u}\\ \quad +\delta^2 2\boldsymbol{\xi}^T\nabla_{\boldsymbol{x}}w\end{array} \end{pmatrix}.$$

One sees from (2.39) that $\gamma = \gamma(\delta\,\boldsymbol{\xi})$ has the expansion

$$\gamma^2 = 1 - \delta^2|\boldsymbol{\xi}|^2 + \delta^4|\boldsymbol{\xi}|^4 - \delta^6|\boldsymbol{\xi}|^6 + \cdots.$$

One can use this and the previous formulae to expand the following quantities:

$$\boldsymbol{\Omega}\boldsymbol{\Omega}^T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & \boldsymbol{\xi} \\ \boldsymbol{\xi}^T & 0 \end{pmatrix} + \delta^2\begin{pmatrix} \boldsymbol{\xi}\boldsymbol{\xi}^T & 0 \\ 0 & -|\boldsymbol{\xi}|^2 \end{pmatrix} + \begin{pmatrix} \mathrm{O}(\delta^4) & \mathrm{O}(\delta^3) \\ \mathrm{O}(\delta^3) & \mathrm{O}(\delta^4) \end{pmatrix}$$

$$\boldsymbol{\Omega}^T\mathbb{D}\boldsymbol{\Omega} = 2\left(\partial_z w + \boldsymbol{\xi}^T\partial_z\boldsymbol{u}\right)$$
$$+\delta^2\left[2\boldsymbol{\xi}^T\nabla_{\boldsymbol{x}}w + \boldsymbol{\xi}^T\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T\right)\boldsymbol{\xi} - 2|\boldsymbol{\xi}|^2\left(\partial_z w + \boldsymbol{\xi}^T\partial_z\boldsymbol{u}\right)\right] + \mathrm{O}(\delta^4)$$

$$\mathbb{D}\boldsymbol{\Omega}\boldsymbol{\Omega}^T + \boldsymbol{\Omega}\boldsymbol{\Omega}^T\mathbb{D} = \delta^{-1}\begin{pmatrix} 0 & \partial_z\boldsymbol{u} \\ \partial_z\boldsymbol{u}^T & 0 \end{pmatrix} + \begin{pmatrix} \partial_z\boldsymbol{u}\boldsymbol{\xi}^T + \boldsymbol{\xi}\partial_z\boldsymbol{u}^T & 0 \\ 0 & 4\partial_z w + 2\boldsymbol{\xi}^T\partial_z\boldsymbol{u} \end{pmatrix}$$
$$+\delta\begin{pmatrix} 0 & \begin{array}{l}(\boldsymbol{\xi}\boldsymbol{\xi}^T - |\boldsymbol{\xi}|^2\boldsymbol{I})\partial_z\boldsymbol{u} + 2\boldsymbol{\xi}\partial_z w\\ \quad +\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T\right)\boldsymbol{\xi} + \nabla_{\boldsymbol{x}}w\end{array} \\ \begin{array}{l}\partial_z\boldsymbol{u}^T(\boldsymbol{\xi}\boldsymbol{\xi}^T - |\boldsymbol{\xi}|^2\boldsymbol{I}) + 2\boldsymbol{\xi}^T\partial_z w\\ \quad +\boldsymbol{\xi}^T\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + (\nabla_{\boldsymbol{x}}\boldsymbol{u})^T\right) + \nabla_{\boldsymbol{x}}^T w\end{array} & 0 \end{pmatrix}$$
$$+\begin{pmatrix} \mathrm{O}(\delta^3) & \mathrm{O}(\delta^2) \\ \mathrm{O}(\delta^2) & \mathrm{O}(\delta^3) \end{pmatrix}.$$

By (A.1)–(A.3) the tensors $\mathbb{D}^H$, $\mathbb{D}^V$ and $\mathbb{D}^E$ therefore have the expansions

$$\mathbb{D}^H = \begin{pmatrix} \nabla_x u + (\nabla_x u)^T - \partial_z u \xi^T - \xi \partial_z u^T + (\partial_z w + \xi^T \partial_z u) I & 0 \\ 0 & 0 \end{pmatrix}$$
$$+\delta \begin{pmatrix} 0 & \begin{matrix} |\xi|^2 \partial_z u - \partial_z w \xi \\ - \left(\nabla_x u + (\nabla_x u)^T\right) \xi \end{matrix} \\ \begin{matrix} |\xi|^2 \partial_z u^T - \partial_z w \xi^T \\ -\xi^T \left(\nabla_x u + (\nabla_x u)^T\right) \end{matrix} & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} O(\delta^2) & O(\delta^3) \\ O(\delta^3) & O(\delta^2) \end{pmatrix}$$

$$\mathbb{D}^V = \frac{1}{\delta} \begin{pmatrix} 0 & \partial_z u \\ \partial_z u^T & 0 \end{pmatrix} + \begin{pmatrix} \partial_z u \xi^T + \xi \partial_z u^T & 0 \\ 0 & -2\xi^T \partial_z u \end{pmatrix}$$
$$+\delta \begin{pmatrix} 0 & \begin{matrix} -(3\xi\xi^T + |\xi|^2 I)\partial_z u - 2\xi \partial_z w \\ + \left(\nabla_x u + (\nabla_x u)^T\right) \xi + \nabla_x w \end{matrix} \\ \begin{matrix} -\partial_z u^T (3\xi\xi^T + |\xi|^2 I) - 2\xi^T \partial_z w \\ +\xi^T \left(\nabla_x u + (\nabla_x u)^T\right) + \nabla_x^T w \end{matrix} & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} O(\delta^2) & O(\delta^3) \\ O(\delta^3) & O(\delta^2) \end{pmatrix}$$

$$\mathbb{D}^E = \left(\partial_z w + \xi^T \partial_z u\right) \left[ \begin{pmatrix} -I & 0 \\ 0 & 2 \end{pmatrix} + 3\delta \begin{pmatrix} 0 & \xi \\ \xi^T & 0 \end{pmatrix} \right] + \begin{pmatrix} O(\delta^2) & O(\delta^3) \\ O(\delta^3) & O(\delta^2) \end{pmatrix}.$$

Upon employing these expansions in the constitutive relation (2.37) we find that the components of the non-dimensional stress tensor have the expansions

$$\mathbb{S}_{xx} = \nu_H^{(0)} \left[ \left(\nabla_x u + (\nabla_x u)^T\right) - \partial_z u \xi^T - \xi \partial_z u^T + \left(\partial_z w + \xi^T \partial_z u\right) I \right] + \nu_V^{(0)} \left[ \partial_z u \xi^T + \xi \partial_z u^T \right] + O(\delta^2) \tag{A.4}$$

$$\mathbb{S}_{xz} = \nu_V^{(0)} \partial_z u + \delta^2 \nu_H^{(0)} \left[ |\xi|^2 \partial_z u - \xi \partial_z w - \left(\nabla_x u + (\nabla_x u)^T\right)\xi \right] + \delta^2 \nu_V^{(0)} \left[ -(3\xi\xi^T + |\xi|^2)\partial_z u - 2\xi \partial_z w + \left(\nabla_x u + (\nabla_x u)^T\right)\xi + \nabla_x w \right] + O(\delta^4) \tag{A.5}$$

$$\mathbb{S}_{zz} = -2\nu_V^{(0)} \left[ \xi^T \partial_z u \right] + O(\delta^2). \tag{A.6}$$

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